CHAPTER 3 SPIN *ET CETERA*

1. Rotations

A rotation is a real linear transformation $x_i \mapsto \sum_j R_{ij} x_j$ of the Cartesian coordinates x_i that leaves invariant the scalar product $\mathbf{x} \cdot \mathbf{y} = \sum_i x_i y_i$. That is

$$\sum_i \left(\sum_j R_{ij} x_j \right) \left(\sum_k R_{ik} y_k \right) = \sum_i x_i y_i \tag{1}$$

By equating coefficients of $x_j y_k$ on both sides of the equation,

$$\sum_i R_{ij} R_{ik} = \delta jk \quad \text{or in matrix notation} \quad R^{\mathrm{T}} R = \mathbb{I} \tag{2}$$

The matrices R are said to be ***orthogonal***.

Taking the facts that ***the determinant of a product of matrices is the product of the determinants***. We see that $[\mathrm{Det}\, R]^2 = 1$, so $\mathrm{Det}\, R$ can only be $+1$ or -1. The set of all real orthogonal matrices forms a group, known as $O(3)$.

Not all transformations $x_i \mapsto \sum_j R_{ij} x_j$ satisfying Eq.2 are rotations. ***The transformations with*** $\mathrm{Det}\, R = -1$ ***are space-inversions.*** The transformations with $\mathrm{Det}\, R = +1$ are rotations, which also forms a group $SO(3)$.

Like other symmetry transformations, a rotation R induces on the Hilbert space of physical states a unitary transformation, in this case $\Psi \mapsto U(R)\Psi$. $U(R)$ must induce a rotation

$$U^{-1}(R) V_i U(R) = \sum_j R_{ij} V_j \tag{3}$$

Rotations can be infinitesimal. In this case,

$$R_{ij} = \delta_{ij} + \omega_{ij} + O(\omega^2) \tag{4}$$

with ω_{ij} infinitesimal. Thus,

$$\mathbb{I} = \left(\mathbb{I} + \omega^{\mathrm{T}} + O(\omega^2) \right) \left(\mathbb{I} + \omega + O(\omega^2) \right) = \mathbb{I} + \omega^{\mathrm{T}} + \omega + O(\omega^2) \tag{5}$$

so $\omega^{\mathrm{T}} = -\omega$, or in other words

$$\omega_{ji} = -\omega_{ij} \tag{6}$$

For such infinitesimal rotations, the unitary operator $U(R)$ must take the form

$$U(\mathbb{I} + \omega) \to \mathbf{1} + \frac{i}{2\hbar} \sum_{ij} \omega_{ij} J_{ij} + O(\omega^2) \tag{7}$$

with $J_{ij} = -J_{ji}$ a set of Hermitian operators.